IDÈLIC CLASS FIELD THEORY FOR 3-MANIFOLDS AND VERY ADMISSIBLE LINKS

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Abstract. We study a topological analogue of idèlic class field theory for 3-manifolds, in the spirit of arithmetic topology. We firstly introduce the notion of a very admissible link $\mathcal{K}$ in a 3-manifold M, which plays a role analogous to the set of primes of a number field. For such a pair $(M,\mathcal{K})$, we introduce the notion of idèles and define the idèle class group. Then, getting the local class field theory for each knot in $\mathcal{K}$ together, we establish analogues of the global reciprocity law and the existence theorem of idèlic class field theory.

1. Introduction

In this paper, following analogies between 3-dimensional topology and number theory, we study a topological analogue of idèlic class field theory for 3-manifolds. We establish topological analogues of Artin's global reciprocity law and the existence theorem of idèlic class field theory, for any closed, oriented, connected 3-manifold.

Let M be a closed, oriented, connected 3-manifold, equipped with a *very admissible link* (knot set) $\mathcal{K}$, the notion introduced in [Nii14] as an analogue of the set of primes in a number ring. For this notion, we give a refined treatment in Section 2.

For $(M,\mathcal{K})$, we also introduce the notion of a *universal $\mathcal{K}$-branched cover* in Section 3, which may be regarded as an analogue of an algebraic closure of a number field. We fix it and restrict our argument to the branched covers which are obtained as its quotients. (It is equivalent to consider isomorphism classes of branched covers with base points.)

We then introduce the *idèle group* $I_{M,\mathcal{K}}$, the *principal idèle group* $P_{M,\mathcal{K}}$, and the *idèle class group* $C_{M,\mathcal{K}} := I_{M,\mathcal{K}}/P_{M,\mathcal{K}}$ in a functorial way. We define $I_{M,\mathcal{K}}$ as the restricted product of $H_1(\partial V_K)$'s where K runs through all the components of $\mathcal{K}$, following [Nii14]. Let $\mathrm{Gal}(X_L^{\mathrm{ab}}/X_L)$ denote the Galois group of the maximal abelian cover over the exterior $X_L = M - L$ of each finite sublink $L \subset \mathcal{K}$, and put $\mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}} := \varprojlim_{L\subset\mathcal{K}} \mathrm{Gal}(X_L^{\mathrm{ab}}/X_L)$. Then there is a natural homomorphism $\widetilde{\rho}_{M,\mathcal{K}} : I_{M,\mathcal{K}} \to \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}}$. In [Nii14], $P_{M,\mathcal{K}}$ was defined as $\mathrm{Ker}\,\widetilde{\rho}_{M,\mathcal{K}}$. In this paper, we define it by the image of the natural homomorphism $\Delta : H_2(M,\mathcal{K}) \to I_{M,\mathcal{K}}$ and prove that it coincides with that of [Nii14] (Theorem 5.3).

The first main result of our idèlic class field theory is the following analogue of Artin's global reciprocity law, which was originally proved in [Nii14] for the case over an integral homology 3-sphere.

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Theorem 5.4 (The global reciprocity law for 3-manifolds)**.** *There is a canonical isomorphism called the global reciprocity map*

$$\rho_{M,\mathcal{K}} : C_{M,\mathcal{K}} \overset{\cong}{\to} \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}}$$

such that (i) *for any finite abelian cover $h : N \to M$ branched over a finite link in $\mathcal{K}$, $\rho_{M,\mathcal{K}}$ induces an isomorphism of finite abelian groups*

$$C_{M,\mathcal{K}}/h_*(C_{N,h^{-1}(\mathcal{K})}) \overset{\cong}{\to} \mathrm{Gal}(h),$$

and (ii) *$\rho_{M,\mathcal{K}}$ is compatible with the local theories.*

In this paper, we also show an analogue of the existence theorem of idèlic class field theory, which gives a bijective correspondence between finite abelian covers of M branched over finite links in $\mathcal{K}$ and certain subgroups of $C_{M,\mathcal{K}}$. For this purpose, we introduce certain topologies on $C_{M,\mathcal{K}}$, called the *standard topology* and the *norm topology*, following after the case of number fields ([KKS11], [Neu99]). Now our existence theorem is stated as follows:

Theorem 7.4 (The existence theorem)**.** *The following correspondence*

$$(h : N \to M) \mapsto h_*(C_{N,h^{-1}(\mathcal{K})})$$

gives a bijection between the set of (isomorphism classes of) finite abelian covers of M branched over finite links L in $\mathcal{K}$ and the set of open subgroups of finite indices of $C_{M,\mathcal{K}}$ with respect to the standard topology. Moreover, the latter set coincides with the set of open subgroups of $C_{M,\mathcal{K}}$ with respect to the norm topology.

Here are contents of this paper. In Section 2, we discuss very admissible links. We refine the definition in [Nii14], prove the existence, and give some remarks. In Section 3, we briefly review the basic analogy between knots and primes, introduce the notion of a universal $\mathcal{K}$-branched cover, and discuss the role of base points. In Section 4, we recall the idèlic class field theory for number fields, whose analogue will be discussed in the later sections. In Section 5, we introduce the idèle group $I_{M,\mathcal{K}}$, the natural homomorphism $\Delta : H_2(M,\mathcal{K}) \to I_{M,\mathcal{K}}$, the principal idèle group $\mathcal{P}_{M,\mathcal{K}}$, and the idèle class group $C_{M,\mathcal{K}}$, together with the global reciprocity map $\rho_{M,\mathcal{K}} : C_{M,\mathcal{K}} \to \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}}$ for a 3-manifold M equipped with a (very admissible) link $\mathcal{K}$. Then we verify the global reciprocity law for 3-manifolds. In Section 6, we introduce the standard topology on the idèle class group $C_{M,\mathcal{K}}$, and prove the existence theorem for it. In Section 7, we do the same for the norm topology. In Section 8, we give some remarks on the norm residue symbols, the class field axiom, and an application to the genus theories.

We note that idèlic class field theory for 3-manifolds was initially studied by A. Sikora ([Sik03], [Sik0s], [Sik11]). In the beginning of our study we were inspired by his work, although our approach presented in this paper is different from his.

Notation and terminology. For a manifold X, we simply denote by $H_n(X)$ its n-th homology group with coefficients in $\mathbb{Z}$. For a group G and its subgroup G_1, we write $G_1 < G$. A knot (resp. a link) in a 3-manifold M means a topological embedding of S^1 (resp. $\sqcup S^1$) into M or its image. A branched cover of a 3-manifold is branched over a link, and is a morphism of spaces equipped with base points outside the branch locus. If $h : N \to M$ is a branched cover of a 3-manifold M and L is a link in M containing a subset of the branch locus only as its connected components,

then $h^{-1}(L)$ denotes the link in N defined by its preimage. When h is Galois (i.e., regular), $\mathrm{Gal}(h) = \mathrm{Deck}(h)$ denotes the Galois group, that is, the group of covering transformations of N over M.

2. Very admissible links

In the previous paper [Nii14], the notion of a very admissible knot set (link) $\mathcal{K}$ in a 3-manifold M was introduced, and was regarded as an analogue object of the set of all the primes in a number field. However, the construction was not sufficient.

In this section, we refine the definitions, prove the existence of a very admissible link $\mathcal{K}$ in any closed, oriented, connected 3-manifold M, and give some remarks.

Idèle theory sums up all the local theories and describes the global theory. In number theory, each prime equips local theory, and in an extension of number field F/k, every prime of F is above some prime of k. In 3-dimensional topology, local theories are the theories of branched covers over the tubular neighborhoods of knots. We define a very admissible link as follows, and regard it as an analogue of the set of all the primes.

Definition 2.1. Let M be a closed, oriented, connected 3-manifold. Let $\mathcal{K}$ be a link in M consisting of countably many (finite or infinite) *tame* components. We say $\mathcal{K}$ is an *admissible link* of M if the components of $\mathcal{K}$ generates $H_1(M)$. We say $\mathcal{K}$ is a *very admissible link* of M if for any finite cover $h : N \to M$ branched over a finite link in $\mathcal{K}$, the components of the link $h^{-1}(\mathcal{K})$ generates $H_1(N)$.

Note that for a link consisting of countably infinite disjoint tame knots, by the Sielpiński theorem ([Eng89, Theorem 6.1.27]), the notion of a *component* makes sense in a natural way, that is, each connected component of its image is the image of some S^1 in the domain.

We may assume that M is a C^∞-manifold. We fix a finite C^∞-triangulation T on M. A knot $K : S^1 \to M$ is called *tame* if it satisfies the following equivalent conditions: (1) There is a self-homeomorphism h of M such that $h(K)$ is a subcomplex of some refinement of T. (2) There is a self-homeomorphism h of M such that $h(K)$ is a C^∞-submanifold of M. (3) There is a tubular neighborhood of K, that is, a topological embedding $\iota_K : S^1 \times D^2 \to M$ with $\iota_K(S^1 \times 0) = K$.

We note that ($\sharp$) if a neighborhood V of K is given, then h in (1) and (2) can be taken so that it has a support in V (i.e., it coincides with id on $M - V$).

proof. (1) $\Longrightarrow$ (2) : We may assume that K itself is a subcomplex of some refinement T' of T. For each 0-simplex v of T' on K, by a self-homeomorphism of M with support in a small neighborhood of v, we can modify K so that K is C^∞ in a neighborhood of v. Doing the similar for every v, we obtain (2).
(2) $\Longrightarrow$ (3): We may assume that K itself is a C^∞-submanifold of M. A tubular neighborhood of V is the total space of a D^2-bundle on $K \cong S^1$. Since M is oriented, V is orientable and hence is the trivial bundle. Hence (3).
(3) $\Longrightarrow$ (1): We use [Moi52, Theorem 5]: Let M be a metrized 3-manifold with a fix triangulation T and let K be a closed subset of M. Suppose that there is a neighborhood V of K in M and a topological embedding $\iota : V \to M$ so that $\iota(K)$ is a subcomplex of a refinement of T. Then, there is a self-homeomorphism $h : M \to M$ such that $h(K)$ is a subcomplex of a refinement of T. In addition, for a given $\varepsilon > 0$, there is some h with its support in the ε-neighborhood of K. Moreover,

we can take h as closer to id as we want $\cdots$ $(*)$. If we apply this theorem to our M with a metric, $T, K, V := \iota_K(S^1 \times D^2)$, and the inclusion ι, then we obtain (1).

By noting $(*)$ and the construction in (1)$\Longrightarrow$ (2), we see ($\sharp$). $\square$

We also remark that (1), (2), (3) are equivalent to that K is *locally flat*, i.e., for each $x \in K$, there is a closed neighborhood V so that $(V, V \cap K)$ is homeomorphic to (D^3, D^1) as pairs ([Moi54, Theorem 8.1], [Bin54, Theorem 9]).

A finite link $L : \sqcup S^1 \hookrightarrow M$ is called *tame* if it satisfies the following equivalent conditions: (1) There is a self-homeomorphism h of M such that $h(L)$ is a subcomplex of some refinement of T. (2) There is a self-homeomorphism h of M such that $h(L)$ is a C^∞-submanifold of M. (3) Each component $K : S^1 \to M$ of L is tame.

The non-trivial part of this equivalence is to prove that (3) implies (1). We can prove it by (3) $\Longrightarrow$ (1) for the knot case and the condition ($\sharp$) on the support of a self-homeomorphism h.

A finite link consisting of tame components always equips a tubular neighborhood as a link. An infinite link L consisting of countably many tame components quips a tubular neighborhood as a link if and only if it has no accumulation point. We do not eliminate the cases with accumulation points.

For a tame knot K in M, we denote a tubular neighborhood by V_K, which is unique up to ambient isotopy. For a link L in M consisting of countably many tame components, we consider *the formal* (or *infinitesimal*) *tubular neighborhood* $V_L := \sqcup_{K \subset L} V_K$, where K runs through all the components of L. *The meridian* $\mu_K \in H_1(\partial V_L)$ of K is the generator of $\mathrm{Ker}(H_1(\partial V_K) \to H_1(V_K))$ corresponding to the orientation of K. A *longitude* $\lambda_K \in H_1(\partial V_L)$ of K is an element satisfying that μ_K and λ_K form a basis of $H_1(\partial V_K)$. We fix a longitude for each K. For a finite branched cover $h : N \to M$ and for each component of $h^{-1}(K)$ in N, we fix a longitude which is a component of the preimage of that of K.

Lemma 2.2. *Let M be a closed, oriented, connected 3-manifold and let L be a link in M consisting of countably many tame components. Then there is a link $\mathcal{L}$ in M containing L, consisting of countably many tame components, and satisfying that for any finite cover $h : N \to M$ branched over a finite sublink of L, $H_1(N)$ is generated by the components of the preimage $h^{-1}(\mathcal{L})$.*

proof. The set of all the finite branched covers of M branched over finite sublinks of L is countable, and can be written as $\{h_i : N_i \to M\}_{i \in \mathbb{N} = \mathbb{N} \cup \{0\}}$, where $h_0 = id_M$. Indeed, for each finite sublink $L' \subset L$, finite branched covers of M branched over L' corresponds to subgroups of $\pi_1(M - L')$ of finite indices. Since $\pi_1(M - L')$ is finitely generated group, such subgroups are countable.

We construct an inclusion sequence $L_0 \subset L_1 \subset \ldots \subset L_i \subset \ldots$ of links consisting of countably many tame components as follows. First, we put $L_0 = L$. Next, for $i \in \mathbb{N}_{>0}$, let L_{i-1} be given. We *claim* that there is a link L_i in M including L_{i-1}, consisting of countably many tame components, and satisfying that the components of the preimage $h_i^{-1}(L_i)$ generates $H_1(N_i)$. By putting $\mathcal{L} := \cup_i L_i$, we obtain an expected link.

The *claim* above can be deduced immediately from the following assertion: *For any finite branched cover $h : N \to M$ and the preimage $\widetilde{L}$ of any link in M consisting of countably many tame components, there is a finite link L' in $N - \widetilde{L}$ consisting of tame components and the image $h(L')$ being also a link.*

Note that N is again a closed, oriented, connected 3-manifold. We may assume that N is a C^∞-manifold. On the space $C^\infty(S^1, N)$ of maps, since S^1 is compact, the well-known two topologies called *the compact open topology (the weak topology)* and *the Whitney topology (the strong topology)* coincide. It is completely metrizable space and satisfies *the Baire property*, that is, *for any countable family of open and dense subsets, their intersection is again dense.* (We refer to [Hir94] for the terminologies and the general facts stated here.)

Let $\{K_j\}_j$ denote the set of components of $\widetilde{L}$. Since $F_j := \{K \in C^\infty(S^1, N) \mid K \cap K_j = \phi\}$ is open and dense, by the Baire property, the intersection $F := \cap_j F_j$ is dense. Put $H_1(N) = \langle a_1, \ldots, a_r \rangle$, and let A_1 denote the set of tame knots $K \in C^\infty(S^1, N)$ satisfying $[K] = a_1$ whose image $h(K)$ in M is also a tame knot. Then A_1 is open and non-empty. Therefore $A_1 \cap F$ is non-empty, and we can take an element K_1' of it. For $1 \leq k \leq r$, if we replace $\widetilde{L}$ by $\widetilde{L} \cup K_1' \cup \ldots \cup K_k'$ and do a similar construction for a_{k+1} successively, then we complete the proof. □

Theorem 2.3. *Let M be a closed, oriented, connected 3-manifold, and L a link in M. Then, there is a very admissible link $\mathcal{K}$ containing L.*

proof. We construct an inclusion sequence of links $\{\mathcal{K}_i\}_i$ as follows: First, we take a link $\mathcal{K}_0$ which includes L and generates $H_1(M)$. Next, for $i \in \mathbb{N}_{>0}$, let $\mathcal{K}_{i-1}$ be given, and let $\mathcal{K}_i$ be a link obtained from $\mathcal{K}_{i-1}$ by Lemma 2.2. Then the union $\mathcal{K} := \cup \mathcal{K}_i$ is a very admissible link. □

Links $\mathcal{L}$ and $\mathcal{K}$ in the lemma and theorem above may be taken smaller than in the constructions. It may be interesting to ask whether they can be finite. Let $M = S^3$. The unknot is very admissible link. If L is the trefoil, by taking branched 2-cover, we see that $\mathcal{K}_1$ is greater than L. We expect that $\mathcal{K}$ has to be infinite. Next, let M be a 3-manifold, and L a minimum admissible link (L can be empty). For an integral homology 3-sphere M, we have $\mathcal{K} = L = \phi$. For a lens space $M = L(p, 1)$ or $M = S^2 \times S^1$, we can take a knot (the core loop) $\mathcal{K} = L = K$.

In the latter sections of this paper, we assume that a very admissible link $\mathcal{K}$ is an infinite link. However, our argument are applicable for finite $\mathcal{K}$ also.

Remark 2.4 (variants)**.** (1) In the definition of a very admissible $\mathcal{K}$ of M, we consider every finite branched covers which are necessarily abelian, so that for each finite abelian branched cover $h : N \to M$, the preimage $h^{-1}(\mathcal{K})$ is also a very admissible link of N. We will discuss a weaker condition on $\mathcal{K}$ in the end of Section 5, **Remark** 5.9.
(2) Let L be an infinite link such that any (ambient isotopy class of) finite link in M is contained in L. There exists such a link. Indeed, since the classes of finite links are countable, by putting links side by side in $S^3 = \mathbb{R}^3 \cup \{\infty\}$, we obtain such a link L, with one limit point at ∞. By using the metric of $\mathbb{R}^3$, we can take a tubular neighborhood of $\mathcal{L}$. If we start the construction form such an infinite link, then we obtain a special very admissible link $\mathcal{K}$, which controls all the finite branched covers of M branched over any finite links in M.

According to [Mor12], counterparts of infinite primes are ends of 3-manifolds. F. Hajir also studies cusps of hyperbolic 3-manifolds as analogues of infinite primes of number fields ([Haj12]). In this paper, since we deal with closed manifolds, the counterpart of the set of infinite primes is empty.

3. The universal $\mathcal{K}$-branched cover

Class field theory deals with all the abelian extension of a number field k in a fixed algebraic closure $\overline{k}$ of k. In this section, we briefly review the analogies between knots and primes. Then, for a 3-manifold M equipped with an infinite (very admissible) link $\mathcal{K}$, we introduce the notion of the universal $\mathcal{K}$-branched cover, which is an analogue of an algebraic closure of a number field. We also discuss the role of base points.

The analogies between knots and primes has been studied systematically by B. Mazur ([Maz64]), M. Kapranov ([Kap95]), A. Reznikov ([Rez97], [Rez00]), M. Morishita ([Mor02], [Mor10], [Mor12]), A. Sikora ([Sik03]) and others, and their research is called arithmetic topology. Here is a basic dictionary of the analogies. For a number field k, let $\mathcal{O}_k$ denote the ring of integer.

3-manifold M	number ring $\operatorname{Spec}\mathcal{O}_k$
knot $K : S^1 \hookrightarrow M$	prime $\mathfrak{p} : \operatorname{Spec}(\mathbb{F}_\mathfrak{p}) \hookrightarrow \operatorname{Spec}\mathcal{O}_k$
link $L = \{K_1, ..., K_r\}$	set of primes $S = \{\mathfrak{p}_1, ..., \mathfrak{p}_r\}$
(branched) cover $h : N \to M$	(ramified) extension F/k
fundamental group $\pi_1(M)$	étale fundamental group $\pi_1^{\text{ét}}(\operatorname{Spec}\mathcal{O}_k)$
$\pi_1(M - L)$	$\pi_1^{\text{ét}}(\operatorname{Spec}\mathcal{O}_k - S)$
1st homology group $H_1(M)$	ideal class group $\mathrm{Cl}(k)$

We put $\overline{\operatorname{Spec}\mathcal{O}_k} = \operatorname{Spec}\mathcal{O}_k \cup \{\text{infinite primes}\}$. There is also an analogy between the Hurewicz isomorphism and the Artin reciprocity in unramified class field theory:

$H_1(M) \cong \mathrm{Gal}(M^{\mathrm{ab}}/M) \cong \pi_1(M)^{\mathrm{ab}}$	$\mathrm{Cl}(k) \cong \mathrm{Gal}(k_{\mathrm{ur}}^{\mathrm{ab}}/k) \cong \pi_1^{\text{ét}}(\overline{\operatorname{Spec}\mathcal{O}_k})^{\mathrm{ab}}$

Here, $M^{\mathrm{ab}} \to M$ and $k_{\mathrm{ur}}^{\mathrm{ab}}/k$ denote the maximal abelian cover and the maximal unramified abelian extension respectively. Moreover, there are branched Galois theories in a parallel manner, where the fundamental groups of the exteriors of knots and primes dominate the branched covers and the ramified extensions respectively. Our project of idèlic class field theory for 3-manifolds aims to pursue the research of analogies in this line. For more analogies, we consult [Uek14], [Uek], [Uek16], [MTTU], and [Nii14] also.

In the following, we discuss an analogue of an/the algebraic closure of a number field. If we say branched covers, unless otherwise mentioned, we consider *branched covers endowed with base points*, that is, we fix base points in all spaces that are compatible with covering maps. For a space X, we denote by b_X the base point.

First, we recall the notion of an isomorphism of branched covers. For covers $h : N \to M$ and $h' : N' \to M$ branched over L, we say they are *isomorphic* (as branched covers endowed with base points) and denote by $h \cong h'$ if there is a (unique) homeomorphism $f : (N, b_N) \overset{\cong}{\to} (N', b_{N'})$ such that $h = h' \circ f$. Let $\underline{h} : Y_L \to X_L$ and $\underline{h}' : Y_L' \to X_L$ denote the restrictions to the exteriors. Then, $h \cong h'$ is equivalent to that $\underline{h}_*(\pi_1(Y_L, b_{Y_L})) = \underline{h}'_*(\pi_1(Y_L', b_{Y_{L'}}))$ in $\pi_1(X_L, b_{X_L})$.

Such notion is extended to the class of branched pro-covers, which are objects obtained as inverse limits of finite branched covers.

Next, we introduce an analogue notion of an algebraic closure of a number field. For a finite link L in a 3-manifold, a branched pro-cover $h_L : \widetilde{M_L} \to M$ is a *universal L-branched cover* of M if it satisfies a certain universality: $h_L : \widetilde{M_L} \to M$ is a minimal object such that any finite cover of M branched over L factor through it. It is unique up to the canonical isomorphisms, and it can be obtained by Fox completion of a universal cover of the exterior $\underline{h_L} : \widetilde{X_L} \to X_L$. (Note that Fox completion is defined for a spread of locally connected T_1-spaces in general. ([Fox57]))

Now, let M be a 3-manifold equipped with an infinite (very admissible) link $\mathcal{K}$. A branched pro-cover $h_{\mathcal{K}} : \widetilde{M_{\mathcal{K}}} \to M$ is a *universal $\mathcal{K}$-branched cover* of M if it satisfies a certain universality: $h_{\mathcal{K}} : \widetilde{M_{\mathcal{K}}} \to M$ is a minimal object such that any finite cover of M branched over a finite link L in $\mathcal{K}$ factor through it.

It can be obtained as the inverse limit of a family of universal L-branched covers, as follows: For each finite link L in $\mathcal{K}$, let $h_L : \widetilde{M_L} \to M$ be a universal L-branched cover of M. By the universality, for each $L \subset L'$, we have a unique map $f_{L,L'} : \widetilde{M_{L'}} \to \widetilde{M_L}$ such that $h_{L'} = h_L \circ f_{L,L'}$. Thus $\{h_L\}_{L \subset \mathcal{K}}$ forms an inverse system. By putting $\widetilde{M_{\mathcal{K}}} = \varprojlim_{L \subset \mathcal{K}} \widetilde{M_L}$, we obtained a universal $\mathcal{K}$-branched cover $h_{\mathcal{K}} : \widetilde{M_{\mathcal{K}}} \to M$ as the composite of the natural map $\widetilde{M_{\mathcal{K}}} \to \widetilde{M_L}$ and h_L.

For the universal $\mathcal{K}$-branched cover, the inverse limit $\pi_1(X_{\mathcal{K}})$ of the fundamental groups of exteriors $\pi_1(X_L)$ $(L \subset \mathcal{K})$ acts on it in a natural way. The finite branched covers of M obtained as quotients of $h_{\mathcal{K}}$ by subgroups of $\pi_1(X_{\mathcal{K}})$ form a complete system of representatives of the isomorphism classes of covers of M branched over links in $\mathcal{K}$.

Therefore, in the latter section of this paper, if we take $(M, \mathcal{K})$, we silently fix a universal $\mathcal{K}$-branched cover, call it "the" universal $\mathcal{K}$-branched cover, and restrict our argument to the branched subcovers obtained as its quotients.

Finally, we discuss an analogue of a base point. The following facts explain the role of base points in branched covers:

Proposition 3.1. *(1) For $(M, \mathcal{K})$, we fix a universal $\mathcal{K}$-branched cover $h_{\mathcal{K}}$. Then, for a branched cover $h : N \to M$ whose base point is forgotten, taking a branched pro-cover $f : \widetilde{M_{\mathcal{K}}} \to N$ such that $h \circ f = h_{\mathcal{K}}$ is equivalent to fixing a base point in N such that $h(b_N) = b_M$.*
(2) Let $h : N \to M$ be a branched cover. Then, a base point of a universal $\mathcal{K}$-branched cover $h_{\mathcal{K}}$ defines a branched pro-cover $f : \widetilde{M_{\mathcal{K}}} \to N$ such that $h_{\mathcal{K}} = h \circ f$.

An analogue of a base point in a 3-manifold is a geometric point of a number field. Let Ω be a sufficiently large field which includes $\mathbb{Q}$, for instance, $\Omega = \mathbb{C}$. Then, for a number field k, choosing a geometric point $x : \operatorname{Spec} \Omega \to \operatorname{Spec} \mathcal{O}_k$ is equivalent to choosing an inclusion $k \hookrightarrow \Omega$. Moreover, choosing base points in a cover $h : N \to M$ which are compatible with the covering map is an analogue of choosing inclusion $k \subset F \hookrightarrow \Omega$ for an extension F/k. For an algebraic closure $\overline{k}/k$ and an extension F/k of a number field k, we have following facts:

Proposition 3.2. *(1) If we fix $\overline{k}/k$ in Ω, taking an inclusion $F \hookrightarrow \overline{k}$ is equivalent to taking an inclusion $F \hookrightarrow \Omega$.*
(2) For an extension F/k in Ω, an inclusion $\overline{k} \hookrightarrow \Omega$ defines $F \hookrightarrow \overline{k}$.

In addition, we have $\operatorname{Spec}\mathcal{O}_k = \{\text{finite primes}\} \cup \operatorname{Spec} k$, and $(\operatorname{Spec} k)(\Omega) = \{\Omega\text{-rational points of} \operatorname{Spec} k\} := \operatorname{Hom}(\operatorname{Spec}\Omega, \operatorname{Spec} k) \cong \operatorname{Hom}(k, \Omega)$. Accordingly, choosing a geometric point (an injection) $k \hookrightarrow \Omega$ is an analogue of choosing a base point in the exterior of $\mathcal{K}$ in M. If $k/\mathbb{Q}$ is Galois, we have a non canonical isomorphism $\{\text{the choices of a geometric point of } k\} = \operatorname{Hom}(k, \Omega) \cong \operatorname{Gal}(k/\mathbb{Q})$. This map depends on the fact that an inclusion of $\mathbb{Q}$ into a field is unique. In order to state an analogue for $(M, \mathcal{K})$, we need to fix an analogue of $k/\mathbb{Q}$. If we fix a Galois branched cover $h_M : M \to S^3$ whose base point is forgotten, an infinite link $\underline{\mathcal{K}}$ in S^3 such that $h^{-1}(\underline{\mathcal{K}}) = \mathcal{K}$, and a base point b_0 in S^3, then we have a non-canonical map $\{\text{the choices of base points in } M\} \cong \operatorname{Gal}(h_M)$.

Thereby, we obtained the following dictionary:

3-manifold with very admissible link $(M, \mathcal{K})$	number ring $\operatorname{Spec}\mathcal{O}_k$
universal $\mathcal{K}$-branched cover $h_{\mathcal{K}} : \widetilde{M_{\mathcal{K}}} \to M$	algebraic closure $\overline{k}/k$
base point $b_M : \{\mathrm{pt}\} \hookrightarrow M$	geometric point $x : \operatorname{Spec}\Omega \to \operatorname{Spec}\mathcal{O}_k$

In this paper, since we consider only regular (Galois) covers, we can forget base points. Then weaker equivalence classes of branched covers should be considered.

4. Idèlic class field theory for number fields

In this section, we briefly review the idèlic class field theory for number fields, whose topological analogues will be studied in later sections. We consult [KKS11] and [Neu99] as basic references for this section.

4.1. Local theory. We firstly review the local theory. Let k be a number field, that is, a finite extension of the rationals $\mathbb{Q}$, and let $\mathfrak{p} \subset \mathcal{O}_k$ be a prime ideal of its integer ring. Then, for a local field $k_{\mathfrak{p}}$, we have the following commutative diagram of splitting exact sequences.

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}_{\mathfrak{p}}^{\times} \arrow[r] \arrow[d] & k_{\mathfrak{p}}^{\times} \arrow[r, "v_{\mathfrak{p}}"] \arrow[d, "\rho_{\mathfrak{p}}"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & \operatorname{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}}^{\mathrm{ur}}) \arrow[r] & \operatorname{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}}) \arrow[r] & \operatorname{Gal}(k_{\mathfrak{p}}^{\mathrm{ur}}/k_{\mathfrak{p}}) \arrow[r] & 1
\end{tikzcd}$$

Here, $\mathcal{O}_{\mathfrak{p}}^{\times}$ is the local unit group, $v_{\mathfrak{p}}$ is the *valuation*, $k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}}$ is the maximal abelian extension, and $k_{\mathfrak{p}}^{\mathrm{ur}}/k_{\mathfrak{p}}$ is the maximal unramified abelian extension. The map $\rho_{\mathfrak{p}}$ is called *the local reciprocity homomorphism*, which is a canonical injective homomorphism with dense image, and controls all the abelian extensions of the local field $k_{\mathfrak{p}}$. In the lower line, $I_{\mathfrak{p}}^{\mathrm{ab}} = \operatorname{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}}^{\mathrm{ur}})$ is the abelian quotient of the inertia group, and we have $\operatorname{Gal}(k_{\mathfrak{p}}^{\mathrm{ur}}/k_{\mathfrak{p}}) \cong \operatorname{Gal}(\overline{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}}) \cong \widehat{\mathbb{Z}}$.

The theory of a local field is rather complicated. There are non-abelian extensions, and there are notions of wild and tame for ramifications. For the tame quotients, we have an exact sequence

$$1 \to I_{\mathfrak{p}}^{t} \to \operatorname{Gal}(\overline{k}_{\mathfrak{p}}/k) \to \operatorname{Gal}(\overline{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}}) \to 1,$$

where $I_{\mathfrak{p}}^{t} = \langle \tau \rangle \cong \prod_{l \neq p} \mathbb{Z}_l$, $\operatorname{Gal}(\overline{k}_{\mathfrak{p}}/k) = \pi_1^t(\operatorname{Spec}(k_{\mathfrak{p}})) = \langle \tau, \sigma \mid \tau^{q-1}[\tau, \sigma] \rangle$, and $\operatorname{Gal}(\overline{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}}) = \langle \sigma \rangle \cong \widehat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$. They are topologically generated by a monodromy τ and the Frobenius σ.

The local theory of an infinite prime $\mathfrak{p} : k \overset{\widetilde{\mathfrak{p}}}{\hookrightarrow} \mathbb{C} \to \mathbb{R}_{\geq 0}; x \mapsto |\widetilde{\mathfrak{p}}(x)|$ is described as follows. If $\mathfrak{p}$ is real, then $v_\mathfrak{p} : k^\times \to \mathbb{R}; x \mapsto \log|\widetilde{\mathfrak{p}}(x)|$ yields an exact sequence $1 \to \{\pm 1\} \to \mathbb{R}^\times \overset{v_\mathfrak{p}}{\to} \mathbb{R} \to 0$. By taking Housdorffication with respect to the local norm topology, we obtain an exact sequence $1 \to \{\pm 1\} \to \{\pm 1\} \to 0 \to 0$. If $\mathfrak{p}$ is complex, then we have an exact sequence $1 \to S^1 \to \mathbb{C}^\times \overset{v_\mathfrak{p}}{\to} \mathbb{R} \to 0$, and obtain an exact sequence $1 \to 1 \to 1 \to 0 \to 0$ of trivial terms in a similar way. We put $\mathcal{O}_\mathfrak{p}^\times = \{\pm 1\}$ or 1 according as $\mathfrak{p}$ is real or complex. In both cases, there are commutative diagrams similar to the case of finite primes.

4.2. **Definitions.** Next, we review the global theory. Let k be a number field. We define the *idèle group* I_k of k by the following restricted product of $k_\mathfrak{p}^\times$ with respect to the local unit groups $\mathcal{O}_\mathfrak{p}^\times$ over all finite and infinite primes $\mathfrak{p}$ of k:

$$I_k := \prod_{\mathfrak{p}}{}' k_\mathfrak{p}^\times = \left\{ (a_\mathfrak{p})_\mathfrak{p} \in \prod_{\mathfrak{p}\text{: prime}} k_\mathfrak{p}^\times \;\middle|\; v_\mathfrak{p}(a_\mathfrak{p}) = 0 \text{ for almost all finite primes } \mathfrak{p} \right\}.$$

This is the restricted products with respect to the local topology on $k_\mathfrak{p}^\times$ (see Subsection 4.3) and the family of open subgroups $\left\{\mathcal{O}_\mathfrak{p}^\times < k_\mathfrak{p}^\times\right\}_\mathfrak{p}$.

Since we have $v_\mathfrak{p}(a) = 0$ for $a \in k^\times$ and for almost all finite primes $\mathfrak{p}$, $k^\times$ is embedded into I_k diagonally. We define the *principal idèle group* P_k of k by the image of the diagonal embedding $\Delta : k^\times \to I_k$, and the *idèle class group* of k by the quotient $C_k := I_k/P_k$.

Then, the homomorphism to the ideal group $\varphi : I_k \to I(k);\ (a_\mathfrak{p})_\mathfrak{p} \mapsto \prod_\mathfrak{p} \mathfrak{p}^{v_\mathfrak{p}(a_\mathfrak{p})}$ induces an isomorphism

$$I_k/(U_k \cdot P_k) \cong \mathrm{Cl}(k),$$

where $U_k = \operatorname{Ker}\varphi = \prod_\mathfrak{p} \mathcal{O}_\mathfrak{p}^\times$ denotes the unit idèle group and $\mathrm{Cl}(k)$ denotes the ideal class group of k.

4.3. **Standard topology.** The idèle class group C_k equips the *standard topology*, which is the quotient topology of the *restricted product topology* on the idèle group I_k of the local topologies, defined as follows.

We firstly consider on $\mathcal{O}_\mathfrak{p}^\times$ the relative topology of *the local norm topology* of $k_\mathfrak{p}^\times$, and re-define the *local topology* on $k_\mathfrak{p}^\times$ as the unique topology such that the inclusion $\mathcal{O}_\mathfrak{p}^\times \hookrightarrow k_\mathfrak{p}^\times$ is open and continuous. Next, for each finite set of primes T which includes all the infinite primes, we consider the product topogy on $G(T) = \prod_{\mathfrak{p}\in T} k_\mathfrak{p}^\times \times \prod_{\mathfrak{p}\notin T} \mathcal{O}_\mathfrak{p}^\times$. Then, we define the *standard topology* on I_k so that each subgroup $H < I_k$ is open if and only if $H \cap G(T)$ is open for every T.

This standard topology on C_k differs from the one defined as the quotient topology of relative topology of product topology of the local topologies on $I_k < \prod k_\mathfrak{p}^\times$, and it is finer than the latter.

4.4. **Norm topology.** For a finite abelian extension F/k, the norm map $N_{F/k} : C_F \to C_k$ is defined as follows.

Let $\mathfrak{p}$ be a prime of k and $F_\mathfrak{p}^\times := \prod_{\mathfrak{P}|\mathfrak{p}} F_\mathfrak{P}^\times$. Each $\alpha_\mathfrak{p} \in F_\mathfrak{p}^\times$ defines a $k_\mathfrak{p}$-linear automorphism $\alpha_\mathfrak{p} : F_\mathfrak{p}^\times \to F_\mathfrak{p}^\times;\ x \mapsto \alpha_\mathfrak{p} x$, and the norm of $\alpha_\mathfrak{p}$ is defined by $N_{F_\mathfrak{p}/k_\mathfrak{p}}(\alpha_\mathfrak{p}) = \det(\alpha_\mathfrak{p})$. It induces a homomorphism $N_{F_\mathfrak{p}/k_\mathfrak{p}} : F_\mathfrak{p}^\times \to k_\mathfrak{p}^\times$, and the norm homomorphism $N_{F/k} : I_F \to I_k$ on the idèle groups. Since $N_{F/k}$ sends

the principal idèles to principal idèles, it also induces the norm homomorphism $N_{F/k}: C_F \to C_k$ on the idèle class groups.

For a number field k, the idèle class group C_k equips the *norm topology*, so that it is a topological group, and the family of $N_{F/k}(C_F)$ is a fundamental system of neighborhoods of 0, where F/k runs through all the finite abelian extensions of k.

Proposition 4.1. *A subgroup H of C_k is open and of finite index with respect to the standard topology if and only if it is open with respect to the norm topology.*

4.5. **Global theory.** Here is a main theorem of idèlic class field theory for number fields (cf. [Neu99], §6, Theorem 6.1):

Theorem 4.2 (Idèlic class field theory for number fields)**.** *Let k be a number field and let k^{ab} denote the maximal abelian extension of k which are fixed in $\mathbb{C}$.*
(1) (Artin's global reciprocity law.) *There is a canonical surjective homomorphism, called the global reciprocity map,*

$$\rho_k : C_k \to \mathrm{Gal}(k^{\mathrm{ab}}/k)$$

which has the following properties:
(i) *For any finite abelian extension F/k in $\mathbb{C}$, ρ_k induces an isomorphism*

$$C_k/N_{F/k}(C_F) \cong \mathrm{Gal}(F/k).$$

(ii) *For each prime $\mathfrak{p}$ of k, we have the following commutative diagram*

$$\begin{tikzcd}
k_{\mathfrak{p}}^{\times} \arrow[r, "\rho_{k_{\mathfrak{p}}}"] \arrow[d, "\iota_{\mathfrak{p}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}}) \arrow[d] \\
C_k \arrow[r, "\rho_k"'] & \mathrm{Gal}(k^{\mathrm{ab}}/k),
\end{tikzcd}$$

where $\iota_{\mathfrak{p}}$ is the map induced by the natural inclusion $k_{\mathfrak{p}}^{\times} \to I_k$.
(2) (The existence theorem.) *The correspondence*

$$F \mapsto \mathcal{N} = N_{F/k}(C_F)$$

gives a bijection between the set of finite abelian extensions F/k in $\mathbb{C}$ and the set of open subgroups $\mathcal{N}$ of finite indices of C_k with respect to the standard topology. Moreover, the latter set coincides with the set of open subgroups of C_k with respect to the norm topology.

In the proof, we use the *norm residue symbol* $(\ ,F/k): C_k \twoheadrightarrow \mathrm{Gal}(F/k)$. For this map, we have $\mathrm{Ker}(\ ,F/k) = N_{F/k}(C_F)$.

5. The global reciprocity law

In this section, we recall the local theory, develop the idèlic class field theory for 3-manifolds with a new definition of the principal idèle group, and present the global reciprocity law over a 3-manifold equipped with a link. We generalize the main result of the previous paper [Nii14], as well as suggest an expansion of the M^2KR-dictionary.

5.1. **Local theory.** Let K be a knot in its tubular neighborhood V_K. In our context, the local theory for 3-manifolds is nothing but the Galois theory for the covers of ∂V_K, which is dominated by an abelian group $\pi_1(\partial V_K) = \langle \mu_K, \lambda_K \mid [\mu_K, \lambda_K]\rangle \cong H_1(\partial V_K) \cong \mathbb{Z}^2$. (In a sense, the tame case of a local field is a "quantized" version of this case.) For each manifold X, let $\mathrm{Gal}(\widetilde{X}/X)$ denote the Galois group of the universal cover. We have the following commutative diagram of exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \langle \mu_K \rangle \arrow[r] \arrow[d] & H_1(\partial V_K) \arrow[r, "v_K"] \arrow[d, "\text{Hur.}"] & H_1(V_K) = \langle \lambda_K \rangle \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Gal}(\widetilde{\partial V_K}/\widetilde{\partial V_K}) \arrow[r] & \mathrm{Gal}(\widetilde{\partial V_K}/\partial V_K) \arrow[r] & \mathrm{Gal}(\widetilde{V_K}/V_K) \arrow[r] & 0
\end{tikzcd}$$

By an isomorphism $\partial_* : H_2(V_K, \partial V_K) = \langle [D_K] \rangle \overset{\cong}{\to} \langle \mu_K \rangle$, classes of meridian disks D_K can be seen as analogues of local units. (An analogue of the unit group of a number field is considered as a surface-related object, while there are several variations of dictionary about units.) The map v_K is the one induced by the natural injection $\partial V_K \hookrightarrow V_K$, which is an analogue of the valuation map $v_{\mathfrak{p}}$ in number theory. The vertical maps are (the inverses of) the Hurewicz isomorphisms. In the lower line, $I_K := \mathrm{Gal}(\widetilde{\partial V_K}/\widetilde{\partial V_K})$ is the inertia group, and we have $\mathrm{Gal}(\widetilde{V_K}/V_K) \cong \mathrm{Gal}(\widetilde{K}/K) \cong \mathbb{Z}$.

5.2. **Definitions.** Let M be a closed, oriented, connected 3-manifold. Let $\mathcal{K}$ be a link in M consisting of countably many tame components with a (the) formal tubular neighborhood $V_{\mathcal{K}} = \sqcup_{K \subset \mathcal{K}} V_K$. For a sublink L of $\mathcal{K}$, we put $V_L = \sqcup_{K \subset L} V_K$.

Definition 5.1 (idèle group)**.** We define the *idèle group* of $(M, \mathcal{K})$ by the restricted product of $H_1(\partial V_K)$ with respect to the subgroups $\{\langle \mu_K \rangle\}_{K \subset \mathcal{K}} = \{\mathrm{Ker}(v_K)\}_{K \subset \mathcal{K}}$:

$$I_{M,\mathcal{K}} := \prod_{K \subset \mathcal{K}}{}' H_1(\partial V_K) = \left\{ (a_K)_K \in \prod_{K \subset \mathcal{K}} H_1(\partial V_K) \;\middle|\; v_K(a_K) = 0 \text{ for almost all } K \right\}.$$

This is the restricted product with respect to the local topology on $H_1(\partial V_K)$ (see Section 6) and the family of open subgroups $\{\langle \mu_K \rangle < H_1(\partial V_K)\}_K$.

The set of finite sublinks of $\mathcal{K}$ is a directed ordered set with respect to the inclusions. In addition, if we take an inclusion sequence $\cdots \subsetneq L_i \subsetneq L_{i+1} \subsetneq \cdots$ of finite sublinks of $\mathcal{K}$ indexed by $i \in \mathbb{N}$, then $\mathcal{K} = \cup_i L_i$ and any finite sublink L of $\mathcal{K}$ is contained in some L_i.

For each finite sublink L of $\mathcal{K}$, we put $X_L = M - L$. Then $H_1(X_L)$'s form an inverse system indexed by $L \subset \mathcal{K}$ with respect to the natural surjections induced by the inclusion maps of the exteriors. By $\varprojlim_L H_1(X_L) \cong \varprojlim_{i \in \mathbb{N}} H_1(X_{L_i})$, the natural projection $\varprojlim_L H_1(X_L) \to H_1(X_L)$ is surjective.

If we put $X_{\mathcal{K}} = M - \mathcal{K}$, then we have $H_1(X_{\mathcal{K}}) \cong \varprojlim_L H_1(X_L)$. Indeed, there is the Milnor exact sequence $0 \to \varprojlim^1_L H_2(X_L) \to H_1(X_{\mathcal{K}}) \to \varprojlim_L H_1(X_L) \to 0$ ([Mil62]). Since $\{H_2(X_L)\}_L$ is a surjective system and satisfies the Mittag-Leffler condition, we have $\varprojlim^1_L H_2(X_L) = 0$.

For each L, let $\mathrm{Gal}(X_L^{\mathrm{ab}}/X_L)$ denote the Galois group of the maximal abelian cover over its exterior X_L. Then $\mathrm{Gal}(X_L^{\mathrm{ab}}/X_L)$'s form an inverse system in a natural

way. We put $\mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}} := \varprojlim_L \mathrm{Gal}(X_L^{\mathrm{ab}}/X_L)$ and regard it as an analogue of $\mathrm{Gal}(k_{\mathrm{ab}}/k)$. We have $\mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}} \underset{\mathrm{Hur}}{\cong} H_1(X_{\mathcal{K}})$.

For a knot K and a finite link L in $\mathcal{K}$, take an ambient isotopy h fixing K and L so that $h(V_K) \subset X_L$ if needed. Then the composite $\partial V_K \overset{\cong}{\to} h(\partial V_K) \hookrightarrow X_L$ with the inclusion induces a natural map $H_1(\partial V_K) \to H_1(X_L)$ commuting with the the Hurewicz maps.

$$\begin{tikzcd}
H_1(\partial V_K) \arrow[r, "\cong", "\mathrm{Hur}"'] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Gal}(\widetilde{\partial V_K}/\partial V_K) \arrow[d] \\
H_1(X_L) \arrow[r, "\cong", "\mathrm{Hur}"'] & \mathrm{Gal}(X_L^{\mathrm{ab}}/X_L)
\end{tikzcd}$$

Let $\rho_{K,L} : H_1(\partial V_K) \to \mathrm{Gal}(X_L^{\mathrm{ab}}/X_L)$ denote their composite, and we consider the map $\rho_L : I_{M,\mathcal{K}} \to \mathrm{Gal}(X_L^{\mathrm{ab}}/X_L) : (a_K)_K \mapsto \sum_{K \subset \mathcal{K}} \rho_{K,L}(a_K)$ where K runs through all the knot in $\mathcal{K}$. This sum makes sense, because it is actually a finite sum for each $(a_K)_K \in I_{M,\mathcal{K}}$, by the definition of the restricted product. Since $(\rho_L)_L$ is compatible with the inverse system, the following homomorphism is induced:

$$\widetilde{\rho}_{M,\mathcal{K}} : I_{M,\mathcal{K}} \to \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}}.$$

If $\mathcal{K}$ is an admissible link, then this map is surjective.

We give a new definition of the principal idèle group and idèle class group by introducing the natural homomorphism $\Delta : H_2(M,\mathcal{K}) \to I_{M,\mathcal{K}}$ in the following.

Let L and L' be finite sublinks of $\mathcal{K}$ with $L \subset L'$. Then the natural surjection $j : C_*(M,L) \twoheadrightarrow C_*(M,L')$ induces the natural injection $j_* : H_2(M,L) \hookrightarrow H_2(M,L')$. We have the natural isomorphism $H_2(M,\mathcal{K}) \overset{\cong}{\to} \varinjlim_{L\subset\mathcal{K}} H_2(M,L)$, where L runs through all the finite sublinks of $\mathcal{K}$ and the transition maps are the natural map j_*'s. In order to prove this, we use the Sielpiński theorem ([Eng89, Theorem 6.1.27]): If a compact Hausdorff connected space X and a countable family $\{X_i\}_{i\in\mathbb{N}}$ of pairwise disjoint closed subsets satisfy $X = \cup_i X_i$, then at most one of X_i is non-empty. By virtue of this theorem, the singular chain groups satisfy $C_n(\mathcal{K}) = \varinjlim_{L\subset\mathcal{K}} C_n(L)$ for each $n \in \mathbb{N}$. The exact sequence $0 \to C_n(L) \to C_n(M) \to C_n(M,L) \to 0$ yields the exact sequence $0 \to C_n(\mathcal{K}) \to C_n(M) \to \varinjlim_{L\subset\mathcal{K}} C_n(M,L) \to 0$. The exact sequence $0 \to C_n(\mathcal{K}) \to C_n(M) \to C_n(M,\mathcal{K}) \to 0$ induces the natural isomorphism $C_n(M,\mathcal{K}) \overset{\cong}{\to} \varinjlim_{L\subset\mathcal{K}} C_n(M,L)$. By taking the long exact sequences and using the five lemma, we obtain the natural isomorphism $H_n(M,\mathcal{K}) \overset{\cong}{\to} \varinjlim_{L\subset\mathcal{K}} H_n(M,L)$.

For each finite sublink L of $\mathcal{K}$, let V'_L be a (usual) tubular neighborhood of L and put $X_L^\circ = M - \mathrm{Int}(V'_L)$. The inclusions $(M,L) \hookrightarrow (M,V'_L)$ and $(X_L^\circ, \partial X_L^\circ) \hookrightarrow (M,V'_L)$ induce isomorphisms $H_2(M,L) \cong H_2(M,V'_L) \cong H_2(X_L^\circ,\partial X_L^\circ)$. We denote by ∂_L the homomorphism $H_2(M,L) \to H_1(\partial V_L)$ given as the composite of $\partial_* : H_2(M,L) \cong H_2(X_L^\circ,\partial X_L^\circ) \to H_1(\partial X_L^\circ)$ and a natural isomorphism $H_1(\partial X_L^\circ) = H_1(\partial V'_L) \overset{\cong}{\to} H_1(\partial V_L)$. We also consider the homomorphism $H_1(\partial V_L) \overset{\cong}{\to} H_1(\partial V'_L) \to H_1(X_L)$. For each finite sublinks L and L' of $\mathcal{K}$ with $L \subset L'$, there is a commutative

diagram

$$\begin{tikzcd} H_2(M,L') \arrow[r, two heads, "\partial_{L'}"] & H_1(\partial V_{L'}) \arrow[d, "\mathrm{pr}"] \\ H_2(M,L) \arrow[u, hook, "j_*"] \arrow[r, two heads, "\partial_L"] \arrow[ur, phantom, "\circlearrowleft"] & H_1(\partial V_L) \end{tikzcd}$$

where pr denotes the projection to the L-components. Thus a natural map from $\varinjlim_{L\subset\mathcal{K}} H_2(M,L)$ to $\varprojlim_{L\subset\mathcal{K}} H_1(\partial V_L) = \prod_{K\subset\mathcal{K}} H_1(\partial V_K)$ is induced. Since longitudinal component does not added by j_*, the image of this map is included in $I_{M,\mathcal{K}}$. Thus we obtain the natural homomorphism $\Delta : H_2(M,\mathcal{K}) \to I_{M,\mathcal{K}}$. If M is a $\mathbb{Q}\mathrm{HS}^3$, then ∂_L is injective for each finite sublink L of $\mathcal{K}$, and hence so is Δ.

Definition 5.2 (Principal idèle group, idèle class group)**.** We define *the principal idèle group* by $P_{M,\mathcal{K}} := \mathrm{Im}(\Delta : H_2(M,\mathcal{K}) \to I_{M,\mathcal{K}})$, and *the idèle class group* by $C_{M,\mathcal{K}} := I_{M,\mathcal{K}}/P_{M,\mathcal{K}}$.

The following assertion tells that our $P_{M,\mathcal{K}}$ coincides with that of [Nii14], and implies the existence of the global reciprocity map $\rho_{M,\mathcal{K}}$ in Theorem 5.4. It gives a topological interpretation of $\mathrm{Ker}\,\widetilde{\rho}_{M,\mathcal{K}}$ and guarantees the correctness of an analogue of the Artin's reciprocity law.

Theorem 5.3. *The equation $P_{M,\mathcal{K}} = \mathrm{Ker}\,\widetilde{\rho}_{M,\mathcal{K}}$ holds, and $\widetilde{\rho}_{M,\mathcal{K}}$ induces a natural isomorphism $\rho_{M,\mathcal{K}} : C_{M,\mathcal{K}} \overset{\cong}{\to} \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}}$.*

proof. The assertion $\mathrm{Im}\,\Delta \subset \mathrm{Ker}\,\widetilde{\rho}_{M,\mathcal{K}}$ holds in a natural way. Indeed, for any $x \in S_{M,\mathcal{K}}$, there is some $L_0 \subset \mathcal{K}$ and some $x_0 \in H_2(M,L_0)$ such that x is the image of x_0 under the natural map $j : H_2(M,L_0) \hookrightarrow H_2(M,\mathcal{K})$. For any finite link L with $L_0 \subset L \subset \mathcal{K}$, there is a commutative diagram

$$\begin{tikzcd} H_2(M,\mathcal{K}) \arrow[r, "\Delta"] & I_{M,\mathcal{K}} \arrow[r, two heads, "\widetilde{\rho}_{M,\mathcal{K}}"] \arrow[d] & \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}} \arrow[r] \arrow[d] & 0 \\ H_2(M,L) \arrow[u, hook, "j"] \arrow[r, "\partial_L"] & H_1(\partial V_L) \arrow[r] & H_1(X_L) \arrow[r] & 0 \end{tikzcd}$$

and the image of x_0 in $H_1(X_L)$ is zero. Thus the image of x in $\mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}}$ is zero, and $\Delta(x) \in \mathrm{Ker}\,\widetilde{\rho}_{M,\mathcal{K}}$ holds.

We prove $\mathrm{Ker}\,\widetilde{\rho}_{M,\mathcal{K}} \subset \mathrm{Im}\,\Delta$ in the following. Let $(a_K) \in \mathrm{Ker}\,\widetilde{\rho}_{M,\mathcal{K}}$. Then there is a finite sublink $L \subset \mathcal{K}$ such that the longitudinal component of (a_K) is zero outside L and that components of L generates $H_1(M)$. Let a denote the image of (a_K) in $H_1(\partial V_L)$. The image of a in $H_1(X_L)$ coincides with that of (a_K) and hence it is zero. By the exact sequence $H_2(M,L) \to H_1(\partial V_L) \to H_1(X_L) \to 0$, there is some $A \in H_2(M,L)$ with $\partial A = a$. We put $(a'_K) = \Delta(j(A))$. Then it is sufficient to prove $(a_K) = (a'_K)$.

Let L' be any finite link with $L \subset L' \subset \mathcal{K}$, and let b and b' denote the images of (a_K) and (a'_K) in $H_1(\partial V_{L'})$ respectively. Then it is sufficient to prove $b = b'$. Note that b' is the image of A under $H_2(M,L) \overset{j_*}{\to} H_2(M,L') \overset{\partial_{L'}}{\to} H_1(\partial V_{L'})$. Now b and b' are both included in $H_1(\partial V_L) \oplus \langle \mu_K \rangle_{K\subset L'-L}$, their images in $H_1(X_{L'})$ are zero, and their images in $H_1(\partial V_L)$ are a.

$$\begin{tikzcd} H_2(M,L') \arrow[r, two heads, "\partial_{L'}"] & H_1(\partial V_{L'}) \arrow[r] \arrow[d, "\mathrm{pr}"] & H_1(X_{L'}) \arrow[d, "\iota_*"] \\ H_2(M,L) \arrow[r, two heads, "\partial_L"] \arrow[u, hook, "j_*"] & H_1(\partial V_L) \arrow[r] & H_1(X_L) \end{tikzcd}$$

We put $c = b' - b$. Then we have $c \in \langle \mu_{L'-L} \rangle$. We regard $Z_2(M,L)$ and $Z_2(M,L')$ as subgroups of $C_2(M)$ with $Z_2(M) \subset Z_2(M,L) \subset Z_2(M,L') \subset C_2(M)$, and denote by ∂ the boundary map on $C_*(M)$. Since the image of c in $H_1(X_{L'})$ is zero, there is some $C \in Z_2(M,L')$ with $\partial_{L'}([C]) = c$.

Let $V'_{L'}$ be a (usual) tubular neighborhood of L'. Then $\partial_* : H_2(M,L') \to H_1(L')$ factors as $H_2(M,L') \overset{\partial_{L'}}{\to} H_1(\partial V_{L'}) \overset{\cong}{\to} H_1(\partial V'_{L'}) \twoheadrightarrow H_1(V'_{L'}) \overset{\cong}{\to} H_1(L')$ with $\langle \mu_K \rangle_{K \subset L'} = \mathrm{Ker}(H_1(\partial V_{L'}) \to H_1(L'))$. Since $\partial_{L'}([C]) \in \langle \mu_K \rangle_{K \subset L'-L}$, we have $\partial_*[C] = 0$, and we can regard $C \in Z_2(M)$.

Let $I : H_2(M) \times H_1(M) \to \mathbb{Z}$ denote *the intersection form* of M. It is a bilinear form defined by counting the intersection points of transversely intersecting representatives with signs. By the universal coefficient theorem, $H_2(M)$ is torsion-free, and I is right-non-degenerate.

Now $H_1(M)$ is generated by components of L by assumption. Since $\partial'_L([C]) \in \mu_{L'-L}$, we have $\partial_L([C]) = 0$ by regarding $C \in Z_2(M,L)$, and each component K_i of L satisfies $I([C],[K_i]) = 0$. This implies $[C] = 0$ and hence $c = \partial_{L'}([C]) = 0$. Therefore we have $b = b'$, and $\Delta : H_2(M,\mathcal{K}) \to \mathrm{Ker}\,\widetilde{\rho}_{M,\mathcal{K}}$ is a surjection. $\square$

Theorem 5.3 expands the M^{2}KR-dictionary as follows, where k is a number field.

1-cycle group $Z_1(M)$	ideal group $I(k)$
$\partial : C_2(M) \to Z_1(M); s \mapsto \partial s$	$(\ ) : k^\times \to I(k); a \mapsto (a)$
1-boundary group $B_1(M) := \mathrm{Im}\,\partial$	principal ideal group $P(k) := \mathrm{Im}\,\partial$
$H_1(M) := Z_1(M)/B_1(M)$	$\mathrm{Cl}(k) := I(k)/P(k)$
idèle group $I_{M,\mathcal{K}}$	idèle group I_k
$\Delta : H_2(M,\mathcal{K}) \to I_{M,\mathcal{K}}$	$\Delta : k^\times \to I_k$
principal idèle group $P_{M,\mathcal{K}} := \mathrm{Im}\,\Delta$	principal idèle group $P_k := \mathrm{Im}\,\Delta$
idèle class group $C_{M,\mathcal{K}} := I_{M,\mathcal{K}}/P_{M,\mathcal{K}}$	idèle class group $C_k := I_k/P_k$

Let $h : N \to M$ be a finite branched cover branched over a finite link L in $\mathcal{K}$. Then the preimage $h^{-1}(\mathcal{K})$ of $\mathcal{K}$ is a link in N, and the covering map h induces the *norm maps* $h_* : I_{N,h^{-1}(\mathcal{K})} \to I_{M,\mathcal{K}}$, $h_* : P_{N,h^{-1}(\mathcal{K})} \to P_{M,\mathcal{K}}$, and $h_* : C_{N,h^{-1}(\mathcal{K})} \to C_{M,\mathcal{K}}$. They satisfy the transitivity (functoriality) in a natural way. If $\mathcal{K}$ is very admissible, then so is $h^{-1}(\mathcal{K})$.

5.3. **The global reciprocity law.** Here is the first part of *the idèlic global class field theory for 3-manifolds and admissible links*, which is the counter part of **Theorem 3.1** (1).

Theorem 5.4 (The global reciprocity law for 3-manifolds)**.** *Let M be a closed, oriented, connected 3-manifold equipped with a very admissible link $\mathcal{K}$. Then, there is a canonical isomorphism called the* global reciprocity map

$$\rho_{M,\mathcal{K}} : C_{M,\mathcal{K}} \overset{\cong}{\to} \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}}$$

which satisfies the following properties:

(i) *For any finite abelian cover $h : N \to M$ branched over a finite link L in $\mathcal{K}$, ρ_M induces an isomorphism*

$$C_{M,\mathcal{K}}/h_*(C_{N,h^{-1}(\mathcal{K})}) \cong \mathrm{Gal}(h).$$

(ii) *For each knot K in $\mathcal{K}$, we have the following commutative diagram:*

$$\begin{tikzcd} H_1(\partial V_K) \arrow[r, "\cong", "\mathrm{Hur}"'] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Gal}(\widetilde{\partial V_K}/\partial V_K) \arrow[d] \\ C_{M,\mathcal{K}} \arrow[r, "\rho_{M,\mathcal{K}}"'] & \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}}, \end{tikzcd}$$

where the vertical maps are induced by the natural inclusions.

Remark 5.5. This theorem refines and generalizes the main result of [Nii14]. We have replaced the definition of $P_{M,\mathcal{K}}$ and hence that of $C_{M,\mathcal{K}}$. In addition, we have removed the assumption that M is an integral homology 3-sphere, that is, $H_1(M) = 0$.

Since $H_1(M)$ is an analogue of $\mathrm{Cl}(k)$ and $\mathrm{Cl}(k)$ is always finite, it may be interesting to restrict the class of M to the rational homology 3-spheres, that is, $H_1(M)$ is finite. There are many results on the class number $\#\,\mathrm{Cl}(k)$ via idèle theory, and our theory may enable us to consider their analogues. (See Subsection 8.3 also.)

The existence of $\rho_{M,\mathcal{K}}$ is done by Lemma 5.3. The compatibility with the local theories (ii) follows from the definition of the map. We give the proof of (i) in the following.

Definition 5.6. We define the *unit idèle group* of $(M,\mathcal{K})$ by the meridian group

$$U_{M,\mathcal{K}} := \{(a_K)_K \in I_{M,\mathcal{K}} \mid v_K(a_K) = 0 \text{ in } H_1(V_K), \text{ for all } K \text{ in } \mathcal{K}\},$$

that is, a subgroup of the "infinite linear combinations" $\sum_{K\subset\mathcal{K}} m_K\mu_K$ $(m_K \in \mathbb{Z})$ of the meridians of $\mathcal{K}$ with $\mathbb{Z}$-coefficients.

Lemma 5.7 (An improvement of [Nii14] Proposition 5.7)**.** *Let M be a closed, oriented, connected 3-manifold equipped with an admissible link $\mathcal{K}$, and L be a finite link in $\mathcal{K}$. We write $U_{M,\mathcal{K}} = U_L \oplus U_{\mathrm{non}L}$, where U_L is the subgroup generated by the meridians of L, and $U_{\mathrm{non}L} := \mathrm{Ker}(\mathrm{pr}_L : U_{M,\mathcal{K}} \twoheadrightarrow U_L)$. Then we have $I_{M,\mathcal{K}}/(P_{M,\mathcal{K}} + U_{\mathrm{non}L}) \cong H_1(X_L)$.*

Especially, if we put $L = \phi$, we have $I_{M,\mathcal{K}}/(P_{M,\mathcal{K}} + U_{M,\mathcal{K}}) \cong H_1(M)$. Moreover, if M is an integral homology 3-sphere, we have $I_{M,\mathcal{K}} = P_{M,\mathcal{K}} \oplus U_{M,\mathcal{K}}$.

Remark 5.8. The assumption in [Nii14, Proposition 5.7] can be paraphrased as follows: $H_1(M)$ is torsion free, and the knots K in $\mathcal{K}$ with non-trivial images in $H_1(M)$ forms its free basis. We succeeded in removing them.

In the proofs, we abbreviate ${}_{M,\mathcal{K}}$ by ${}_M$, and ${}_{N,h^{-1}(\mathcal{K})}$ by ${}_N$ for simplicity.

proof. For a map $\varphi_L : I_M \twoheadrightarrow H_1(X_L)$, we prove $\mathrm{Ker}\,\varphi_L = P_M + U_{\mathrm{non}L}$. Consider the composite $\varphi_L : I_M \twoheadrightarrow I_M/P_M = C_M \cong \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}} \cong \varprojlim_{L'} H_1(X_{L'}) \twoheadrightarrow H_1(X_L)$. For each $L \subset L' \subset \mathcal{K}$, it factorizes as $\varphi_L : I_M \underset{\varphi_{L'}}{\twoheadrightarrow} H_1(X_{L'}) \underset{pr}{\twoheadrightarrow} H_1(X_L)$. For the meridian μ_K of K in I_M, the Mayer–Vietoris exact sequence proves $N_{L'} := \mathrm{Ker}(\mathrm{pr} : H_1(X_{L'}) \twoheadrightarrow H_1(X_L)) = \langle \varphi_{L'}(\mu_K) \mid K \subset L' - L\rangle$. Hence $\mathrm{Ker}(I_M/P_M \twoheadrightarrow H_1(X_L)) = U_{\mathrm{non}L} \bmod P_M \cong \varprojlim_{L'} N_{L'} = \mathrm{Ker}(\varprojlim_{L'} H_1(X_{L'}) \twoheadrightarrow H_1(X_L))$, and therefore $\mathrm{Ker}\,\varphi_L = P_M + U_{\mathrm{non}L}$. $\square$

proof of Theorem 5.4 (i). Since there are isomorphisms

$$C_M/h_*(C_N) \cong (I_M/P_M)/h_*(I_N/P_N) \cong I_M/(P_M + h_*(I_N)),$$

we consider the natural surjection $\varphi' : I_M \underset{\varphi_L}{\twoheadrightarrow} H_1(X_L) \twoheadrightarrow H_1(X_L)/h_*(H_1(Y_L))$. Since $\mathcal{K}$ is very admissible, there is a surjection $I_N \twoheadrightarrow H_1(Y_L)$, and hence a surjection $h_*(I_N) \twoheadrightarrow h_*(H_1(Y_L))$. Then, there is the following commutative diagram.

$$\begin{tikzcd} h_*(I_N) \arrow[r, two heads] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & h_*(H_1(Y_L)) \arrow[d, hook] \\ I_M \arrow[r, two heads, "\varphi_L"'] & H_1(X_L) \end{tikzcd}$$

Since $\operatorname{Ker}\varphi_L = P_M + U_{\mathrm{non}L} < P_M + h_*(I_N)$, we have $\operatorname{Ker}\varphi' = \operatorname{Ker}\varphi_L + h_*(I_N) = P_M + h_*(I_N)$, and hence $I_M/(P_M + h_*(I_N)) \cong H_1(X_L)/h_*(H_1(Y_L)) \cong \mathrm{Gal}(h)$. $\square$

Remark 5.9. In the proof above, we used the assumption that "$\mathcal{K}$ is very admissible" only to say that $I_{N,h^{-1}(\mathcal{K})} \to h_*(H_1(Y_L))$ is surjective. Therefore, in the global reciprocity law (Theorem 5.4), we can replace the assumption on $\mathcal{K}$ by a weaker (and necessary) one: "for any finite abelian branched cover $h : N \to M$, the natural map $I_{N,h^{-1}(\mathcal{K})} \to h_*(H_1(N))$ is surjective". Especially, if M is an integral homology 3-sphere, then it becomes the empty condition.

6. The standard topology and the existence theorem 1/2

In this section, we introduce the standard topology on the idèle class group of a 3-manifold, and prove the existence theorem 1/2.

Let M be a closed, oriented, connected 3-manifold equipped with a very admissible link $\mathcal{K}$. For each group $\pi_1(\partial V_K) \cong H_1(\partial V_K) = \langle \mu_K \rangle \oplus \langle \lambda_K \rangle \cong \mathbb{Z}^{\oplus 2}$ of the boundary of a tubular neighborhood of each knot K in $\mathcal{K}$, we define an analogue of the local topology of $k_{\mathfrak{p}}^{\times}$. Here μ_K and λ_K denote the meridian and the fixed longitude of K respectively. We first consider the *local norm topology* on $H_1(\partial V_K)$, whose open subgroups correspond to the finite abelian covers of ∂V_K. This topology is equal to the *Krull topology*, whose open subgroups are the subgroups of finite indices. Then we consider the relative topology on the local inertia group $\langle \mu_K \rangle < H_1(\partial V_K)$, and re-define the *local topology* on $H_1(\partial V_K)$ as the unique topology such that the inclusion $\iota : \langle \mu_K \rangle \hookrightarrow H_1(\partial V_K)$ is open and continuous. For this topology, under the identification $\mathbb{Z} \cong \langle \mu_K \rangle \hookrightarrow H_1(\partial V_K) = \langle \mu_K \rangle \oplus \langle \lambda_K \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, the open subgroup of $H_1(\partial V_K)$ has the form $\langle (a, 0), (b, c) \rangle$ with some $a, b, c \in \mathbb{Z}$, $a \neq 0$. Then, the local existence theorem is stated as the 1-1 correspondence between the open subgroups of finite indices and the finite abelian covers.

With this local topology, $I_{M,\mathcal{K}}$ is the restricted product with respect to the open subgroups $\langle \mu_K \rangle < H_1(\partial V_K)$, and $I_{M,\mathcal{K}}$ equips the *restricted product topology* as follows. For each finite link L in $\mathcal{K}$, let $G(L) := \prod_{K \subset L} H_1(\partial V_K) \times \prod_{K \not\subset L} \langle \mu_K \rangle$, and consider the product topology on $G(L)$. Then a subgroup $H < I_{M,\mathcal{K}}$ is open if and only if $H \cap G(L)$ is open for every L.

Definition 6.1. We endow $C_{M,\mathcal{K}}$ with the quotient topology of the restricted product topology of $I_{M,\mathcal{K}}$ and call it the *standard topology*.

The restricted product topology on $I_{M,\mathcal{K}}$ is finer than the relative topology of the product topology of the local topologies, while open subgroups of finite indices coincide. We consider the former in the following.

We study a factorization of $I_{M,\mathcal{K}} \twoheadrightarrow C_{M,\mathcal{K}}$ which helps us to deal with open subgroups of $C_{M,\mathcal{K}}$. We fix a finite sublink L_0 of $\mathcal{K}$ whose components generate $H_1(M)$. For each sublink $L \subset \mathcal{K}$, we put $J_L := \prod_{K\subset L_0} H_1(\partial V_K) \times \prod_{K\subset L-L_0} \langle \mu_K \rangle$. Note that $J_{\mathcal{K}} = G(L_0)$ is an open subgroup of $I_{M,\mathcal{K}}$.

For finite sublinks L and L' with $L_0 \subset L \subset L' \subset \mathcal{K}$, the natural maps form the following commutative diagram.

$$\begin{tikzcd} J_{L'} \arrow[r, two heads] \arrow[d, "\mathrm{pr}"'] \arrow[dr, phantom, "\circlearrowright"] & H_1(X_{L'}) \arrow[d] \\ J_L \arrow[r, two heads] & H_1(X_L) \end{tikzcd}$$

The natural map $\mathrm{Ker}(J'_L \twoheadrightarrow H_1(X_{L'})) \to \mathrm{Ker}(J_L \twoheadrightarrow H_1(X_L))$ is surjective. Indeed, let $x \in \mathrm{Ker}(J_L \twoheadrightarrow H_1(X_L))$ and let x also denote its image by $J_L \hookrightarrow J_L \oplus \prod_{K\subset L'-L} \langle \mu_K \rangle = J_{L'}; x \mapsto x+0$. Since $\mathrm{Ker}(H_1(X'_L) \twoheadrightarrow H_1(X_L))$ is generated by the meridians of $L'-L$, There is some $a \in \prod_{K\subset L'-L}\langle \mu_K\rangle$ such that the images of x and a in $H_1(X_{L'})$ coincide. If we put $y = x-a$, then $y \in \mathrm{Ker}(J'_L \twoheadrightarrow H_1(X_{L'}))$ and its image in J_L is x. Since $\{\mathrm{Ker}(J_L \twoheadrightarrow H_1(X_L))\}_L$ forms a surjective system and satisfies the Mittag-Leffler condition, we have a natural surjection $J_{\mathcal{K}} = \varprojlim_{\mathrm{pr},L} J_L \twoheadrightarrow C_{M,\mathcal{K}}$.

For each knot K' in $\mathcal{K}$ with $K' \not\subset L_0$, we take an element $x_{K'} \in J_{\mathcal{K}}$ satisfying $\lambda_{K'} - x_{K'} \in P_{M,\mathcal{K}} = \ker \rho_{M,\mathcal{K}}$. Put $Q := \langle \lambda_{K'} - x_{K'} \mid K' \not\subset L_0 \rangle < I_{M,\mathcal{K}}$. Then $J_{\mathcal{K}} \hookrightarrow I_{M,\mathcal{K}} \twoheadrightarrow C_{M,\mathcal{K}}$ factors through $I' := I_{M,\mathcal{K}}/Q \cong (\prod_{K\subset\mathcal{K}} \mathbb{Z}) \times (\prod_{K\subset L_0} \mathbb{Z})$.

Let I' be endowed with the quotient topology of the standard topology of $I_{M,\mathcal{K}}$. Since $J_{\mathcal{K}}$ is open, the induced group isomorphism $J_{\mathcal{K}} \overset{\cong}{\to} I'$ is a homeomorphism.

Proposition 6.2. *Let $C_{M,\mathcal{K}}$ be endowed with the standard topology. If M is a rational homology 3-sphere, then an open subgroup of $C_{M,\mathcal{K}}$ is of finite index.*

proof. Put $P' = \mathrm{Ker}(I' \twoheadrightarrow C_{M,\mathcal{K}})$. Then we have $I'/(\prod_K \langle \mu_K \rangle + P') \cong H_1(M)$. The assumption on M means that $H_1(M)$ is a finite group, and hence $\prod_K \langle \mu_K \rangle + P' < I'$ is of finite index. Recall $G(L_0) = J_{\mathcal{K}} \cong I'$ as topological groups. If V is an open subgroup of I', then $V \cap \prod_K \langle \mu_K \rangle < \prod_K \langle \mu_K \rangle$ is of finite index. Let U be an open subgroup of $C_{M,\mathcal{K}}$ and let V denote the preimage of U in I'. Then V is an open subgroup of I' containing P'. Therefore $V < I'$ is of finite index, and so is $U < C_{M,\mathcal{K}}$. □

Theorem 6.3 (The existence theorem 1/2)**.** *Let $C_{M,\mathcal{K}}$ be endowed with the standard topology. Then the correspondence*

$$(h : N \to M) \mapsto h_*(C_{N,h^{-1}(\mathcal{K})})$$

gives a bijection between the set of (isomorphism classes of) finite abelian covers of M branched over finite links L in $\mathcal{K}$ and the set of open subgroups of finite indices of $C_{M,\mathcal{K}}$ with respect to the standard topology.

proof. For each finite link L with $L_0 \subset L \subset \mathcal{K}$, let Cov_L denote the set of finite abelian covers $h : N \to M$ branched over sublinks of L, and let $\mathcal{O}_L$ denote the set of open subgroups of $C_{M,\mathcal{K}}$ of finite indices containing $\mathrm{Ker}(C_{M,\mathcal{K}} \twoheadrightarrow H_1(X_L))$.

Let U be an open subgroup of $C_{M,\mathcal{K}}$ of finite index and let V denote the preimage of U by $I' \twoheadrightarrow C_{M,\mathcal{K}}$. Since V is an open subgroup of I' of finite index, there is some finite link L with $L_0 \subset L \subset \mathcal{K}$ such that V contains a subgroup $\prod_{K \not\subset L} \langle \mu_K \rangle \times \prod_{K \subset L} a_K \langle \mu_K \rangle$ ($a_K \in \mathbb{N}$) and hence contains $\prod_{K \not\subset L} \langle \mu_K \rangle \times \prod_{K \subset L} 0$. Therefore, U contains the image of $\prod_{K \not\subset L} \langle \mu_K \rangle \times \prod_{K \subset L} 0$, which coincides with $\mathrm{Ker}(C_{M,\mathcal{K}} \twoheadrightarrow H_1(X_L))$. Thus the union $\cup_L \mathcal{O}_L$ coincides with the set of all the open subgroups of $C_{M,\mathcal{K}}$ of finite indices.

Conversely, if U is a subgroup of $C_{M,\mathcal{K}}$ of finite index containing $\mathrm{Ker}(C_{M,\mathcal{K}} \twoheadrightarrow H_1(X_L))$ for a finite link L with $L_0 \subset L \subset \mathcal{K}$, then U is open.

For each finite link L with $L_0 \subset L \subset \mathcal{K}$, we have a natural bijection $\mathrm{Cov}_L \to \mathcal{O}_L$ by the Galois correspondence. In addition, for each finite links L and L' with $L_0 \subset L \subset L' \subset \mathcal{K}$, the inclusions $\mathrm{Cov}_L \subset \mathrm{Cov}_{L'}$ and $\mathcal{O}_L \subset \mathcal{O}'_L$ are compatible with the Galois correspondences.

The union $\cup_L \mathrm{Cov}_L$ is the set of all the finite abelian covers branched over finite sublinks of $\mathcal{K}$. Since the inductive limit of bijective maps is again bijective, we obtain the desired bijection. □

7. The norm topology and the existence theorem

In this section, we introduce the *norm topology* on the idèle class group, and present the existence theorem.

Let M be a closed, oriented, connected 3-manifold equipped with a very admissible link $\mathcal{K}$ as before. In the proofs, we use the abbreviations $C_M = C_{M,\mathcal{K}}$ and $C_N = C_{N,h^{-1}(\mathcal{K})}$ for a branched cover $h : N \to M$.

Definition 7.1. We define the *norm topology* on C_M to be the topology of topological group generated by the family $\mathcal{V} := \{h_*(C_{N,h^{-1}(\mathcal{K})})\}$, where $h : N \to M$ runs through all the finite abelian covers of M branched over finite links in $\mathcal{K}$.

Lemma 7.2. *$\mathcal{V}$ is a fundamental system of neighborhoods of 0.*

proof. For any $V_1, V_2 \in \mathcal{V}$, it is suffice to prove $\exists V_3 \in \mathcal{V}$ such that $V_3 \subset V_1 \cap V_2$. However, we prove $V_3 := V_1 \cap V_2 \in \mathcal{V}$.

Let $h_i : N_i \to M$ be a finite abelian cover branched over L_i in $\mathcal{K}$ for $i = 1, 2$. Let $L := L_1 \cup L_2$, and let $G_L := \mathrm{Gal}(X_L^{\mathrm{ab}}/X_L)$ denote the Galois group of the maximal abelian cover over the exterior $X_L = M \setminus \mathrm{Int}(V_L)$. Then, if a cover $h : N \to M$ is unbranched outside L, the map $C_M \twoheadrightarrow \mathrm{Gal}(h)$ factors through the natural map $\varphi_L : C_M \twoheadrightarrow G_L$.

Let $G_i := \mathrm{Ker}(G_L \twoheadrightarrow \mathrm{Gal}(h_i)) < G_L$ for $i = 1, 2$, and let $G_3 := G_1 \cap G_2$. Since G_3 is also a subgroup of G_L of finite index, the ordinary Galois theory for branched covers gives a cover $h_3 : N_3 \to M$ such that $G_3 = \mathrm{Ker}(G_L \twoheadrightarrow \mathrm{Gal}(h_3))$. (This cover h_3 should be called the "composition cover" of h_1 and h_2, because it is an analogue of the composition field k_1k_2 of k_1 and k_2 in number theory.)

Now, Theorem 5.4 (the global reciprocity law) implies $h_{i*}(C_{N_i}) = \varphi_L^{-1}(G_i)$ for $i = 1, 2, 3$, and therefore $h_{3*}(C_{N_3}) = \varphi_L^{-1}(G_3) = \varphi_L^{-1}(G_1 \cap G_2) = \varphi_L^{-1}(G_1) \cap \varphi_L^{-1}(G_2) = h_{1*}(C_{N_1}) \cap h_{2*}(C_{N_2})$. □

Proposition 7.3. *Let $C_{M,\mathcal{K}}$ be endowed with the norm topology. A subgroup V of $C_{M,\mathcal{K}}$ is open if and only if it is closed and of finite index.*

proof. Let V be an open subgroup of C_M. The coset decomposition of C_M by V proves that V is closed. Lemma 7.2 gives a finite abelian branched cover $h : N \to M$ such that $h_*(C_N) < V$. Then Theorem 5.4 implies $(h_*(C_N) : V)(V : C_M) = (h_*(C_N) : C_M) = \#\,\mathrm{Gal}(h)$, and hence V is of finite index.

The converse is also clear by the coset decomposition. □

Now we present the existence theorem for 3-manifolds with respect to both the standard topology and the norm topology, which is the counter part of **Theorem 4.2** (2).

Theorem 7.4 (The existence theorem)**.** *Let M be a closed, oriented, connected 3-manifold equipped with a very admissible link $\mathcal{K}$. Then the correspondence*

$$(h : N \to M) \mapsto h_*(C_{N,h^{-1}(\mathcal{K})})$$

gives a bijection between the set of (isomorphism classes of) finite abelian covers of M branched over finite links L in $\mathcal{K}$ and the set of open subgroups of finite indices of $C_{M,\mathcal{K}}$ with respect to the standard topology. Moreover, the latter set coincides with the set of open subgroups of $C_{M,\mathcal{K}}$ with respect to the norm topology.

proof. The former part is done by Theorem 6.3. We prove the theorem for the norm topology. For a finite abelian cover $h : N \to M$ branched over a finite link in $\mathcal{K}$, the isomorphism $C_M/h_*(C_N) \cong \mathrm{Gal}(h)$ in Theorem 5.4 (the global reciprocity law) gives the following bijections.

$$\begin{aligned}\{C' \mid h_*(C_N) < C' < C_M\} &\longleftrightarrow \{H \mid H < C_M/h_*(C_N) \cong \mathrm{Gal}(h)\}\\ &\longleftrightarrow \{\text{subcovers of } h\}\end{aligned}$$

(Injectivity) For covers h_1 and h_2, this bijections proves that $h_{1*}(C_{N_1}) < h_{2*}(C_{N_2})$ $\underset{\text{iff}}{\Longleftrightarrow}$ h_2 is a subcover of h_1, and hence $h_{1*}(C_{N_1}) = h_{2*}(C_{N_2})$ $\underset{\text{iff}}{\Longleftrightarrow}$ $h_2 = h_1$.
(Surjectivity) For an open subgroup $C' < C_M$, Lemma 7.2 gives a cover $h : N \to M$ such that $h_*(C_N) < C'$, and then the above bijection gives a cover h' which corresponds to C'. □

Corollary 7.5. *If M is a rational homology 3-sphere, the standard topology and the norm topology on $C_{M,\mathcal{K}}$ coincide.*

proof. By Proposition 6.2, it follows immediately from the existence theorem. □

8. Remarks

8.1. **Norm residue symbols.** In the proof of the existence theorem for number fields (Theorem 4.2 (2)), the norm residue symbol played an essential role ([Neu99]).

Let M be a closed, oriented, connected 3-manifold equipped with a very admissible knot set $\mathcal{K}$. For a finite abelian cover $h : N \to M$ branched over a finite link L in $\mathcal{K}$, we define the *norm residue symbol* $(\ ,h) : C_{M,\mathcal{K}} \twoheadrightarrow \mathrm{Gal}(h)$ by the composite of $\rho_{M,\mathcal{K}} : C_{M,\mathcal{K}} \twoheadrightarrow \mathrm{Gal}(M^{\mathrm{ab}}/M)$ and $\mathrm{Gal}(M^{\mathrm{ab}}/M) \twoheadrightarrow \mathrm{Gal}(h)$. For this map, we have $\mathrm{Ker}(\ ,h) = h_*(C_{N,h^{-1}(\mathcal{K})})$.

By using the norm residue symbol for 3-manifolds, we can give another proof for the existence theorem for 3-manifolds (Theorem 7.4).

The norm reside symbols are extensions of the Legendre symbol and the linking number (mod 2). In number theory, the quadratic reciprocity law was deduced from the global reciprocity law ([KKS11, Chapter 5]). In a similar way, we can

deduce the symmetricity of the linking number (mod 2) from our global reciprocity law (Theorem 5.4 (1)).

8.2. **The class field axioms.** The axiom of class field theory ([Neu99]) does not hold for our modules: Let M be an closed, oriented, connected 3-manifold equipped with a very admissible link $\mathcal{K}$, $h : N \to M$ a cyclic branched cover of degree n branched over some $L \subset \mathcal{K}$, and put $G = \mathrm{Gal}(h)$. Then the Tate cohomologies do not necessarily satisfy the following: (i) $\widehat{H}^0(G, C_{N,h^{-1}(\mathcal{K})}) \cong \mathbb{Z}/n\mathbb{Z}$, (ii) $\widehat{H}^1(G, C_{N,h^{-1}(\mathcal{K})}) = 0$, (iii) $\widehat{H}^i(G, U_{N,h^{-1}(\mathcal{K})}) = 0$.

It is recently announced by Mihara that he gave another formulation of idelic class field theory satisfying the axiom of class field theory for 3-manifolds equipped with our very admissible links, compatible with our work, by introducing the notion of *the finite étale cohomology* of 1-cocycle sheaves ([Mih16]).

8.3. **Application to the genus theory.** The genus formula for finite Galois extensions of number fields was given by Furuta with use of idèle theory ([Fur67]). In [Uek16], the second author formulated its analogue for finite branched Galois covers over $\mathbb{Q}\mathrm{HS}^3$'s, gave a parallel proof to the original one by using our idèle theory, and generalized Morishita's work ([Mor01]).

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